0962CH12

CHAPTER 12

HERON'S FORMULA

12.1 Introduction

You have studied in earlier classes about figures of different shapes such as squares, rectangles, triangles and quadrilaterals. You have also calculated perimeters and the areas of some of these figures like rectangle, square etc. For instance, you can find the area and the perimeter of the floor of your classroom.

Let us take a walk around the floor along its sides once; the distance we walk is its perimeter. The size of the floor of the room is its area.

So, if your classroom is rectangular with length 10 m and width 8 m, its perimeter would be 2(10 m + 8 m) = 36 m and its area would be 10 m × 8 m, i.e., 80 m^2.

Unit of measurement for length or breadth is taken as metre (m) or centimetre (cm) etc.

Unit of measurement for area of any plane figure is taken as square metre (m^2) or square centimetre (cm^2) etc.

Suppose that you are sitting in a triangular garden. How would you find its area? From Chapter 9 and from your earlier classes, you know that:

$$\textbf{Area of a triangle} = \frac{1}{2} \times \textbf{base} \times \textbf{height} \qquad \text{(I)}$$

We see that when the triangle is **right angled**, we can directly apply the formula by using two sides containing the right angle as base and height. For example, suppose that the sides of a right triangle ABC are 5 cm, 12 cm and 13 cm; we take base as 12 cm and height as 5 cm (see Fig. 12.1). Then the

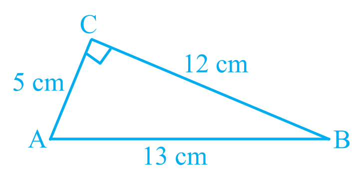

Fig. 12.1

area of Δ ABC is given by

$$\frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 12 \times 5 \text{ cm}^2, \text{ i.e., } 30 \text{ cm}^2$$

Note that we could also take 5 cm as the base and 12 cm as height.

Now suppose we want to find the area of an **equilateral triangle** PQR with side 10cm (see Fig. 12.2). To find its area we need its height. Can you find the height of this triangle?

Let us recall how we find its height when we know its sides. This is possible in an equilateral triangle. Take the mid-point of QR as M and join it to P. We know that PMQ is a right triangle. Therefore, by using Pythagoras Theorem, we can find the length PM as shown below:

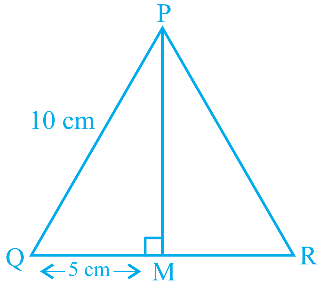

Fig. 12.2

$$PQ^2 = PM^2 + QM^2$$

i.e., $(10)^2 = PM^2 + (5)^2$, since QM = MR.

Therefore, we have $PM^2 = 75$

i.e., $PM = \sqrt{75}$ cm $= 5\sqrt{3}$ cm.

Then area of Δ PQR $= \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 10 \times 5\sqrt{3} \text{ cm}^2 = 25\sqrt{3} \text{ cm}^2$.

Let us see now whether we can calculate the area of an **isosceles triangle** also with the help of this formula. For example, we take a triangle XYZ with two equal sides XY and XZ as 5 cm each and unequal side YZ as 8 cm (see Fig. 12.3).

In this case also, we want to know the height of the triangle. So, from X we draw a perpendicular XP to side YZ. You can see that this perpendicular XP divides the base YZ of the triangle in two equal parts.

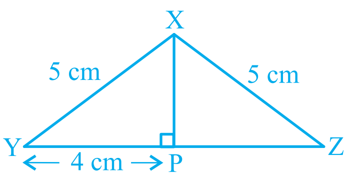

Fig. 12.3

Therefore, $YP = PZ = \frac{1}{2} YZ = 4$ cm

Then, by using Pythagoras theorem, we get

$$XP^2 = XY^2 - YP^2$$

$$= 5^2 - 4^2 = 25 - 16 = 9$$

So, XP = 3 cm

Now, area of Δ XYZ $= \frac{1}{2} \times \text{base YZ} \times \text{height XP}$

$$= \frac{1}{2} \times 8 \times 3 \text{ cm}^2 = 12 \text{ cm}^2.$$

Now suppose that we know the lengths of the sides of a scalene triangle and not the height. Can you still find its area? For instance, you have a triangular park whose sides are 40 m, 32 m, and 24 m. How will you calculate its area? Definitely if you want to apply the formula, you will have to calculate its height. But we do not have a clue to calculate the height. Try doing so. If you are not able to get it, then go to the next section.

12.2 Area of a Triangle — by Heron's Formula

Heron was born in about 10AD possibly in Alexandria in Egypt. He worked in applied mathematics. His works on mathematical and physical subjects are so numerous and varied that he is considered to be an encyclopedic writer in these fields. His geometrical works deal largely with problems on mensuration written in three books. Book I deals with the area of squares, rectangles, triangles, trapezoids (trapezia), various other specialised quadrilaterals, the regular polygons, circles, surfaces of cylinders, cones, spheres etc. In this book, Heron has derived the famous formula for the area of a triangle in terms of its three sides.

Heron (10 C.E. – 75 C.E.)

Fig. 12.4

The formula given by Heron about the area of a triangle, is also known as *Hero's formula*. It is stated as:

$$\textbf{Area of a triangle} = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{(II)}$$

where a, b and c are the sides of the triangle, and s = *semi-perimeter, i.e., half the perimeter of the triangle* = $\dfrac{a+b+c}{2}$,

This formula is helpful where it is not possible to find the height of the triangle easily. Let us apply it to calculate the area of the triangular park ABC, mentioned above (see Fig. 12.5).

Let us take a = 40 m, b = 24 m, c = 32 m,

so that we have $s = \dfrac{40 + 24 + 32}{2}$ m = 48 m.

$s - a = (48 - 40)$ m = 8 m,

$s - b = (48 - 24)$ m = 24 m,

$s - c = (48 - 32)$ m = 16 m.

Therefore, area of the park ABC

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{48 \times 8 \times 24 \times 16}\ \text{m}^2 = 384\ \text{m}^2$$

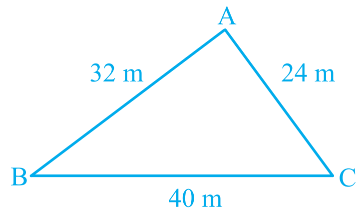

Fig. 12.5

We see that $32^2 + 24^2 = 1024 + 576 = 1600 = 40^2$. Therefore, the sides of the park make a right triangle. The largest side, i.e., BC which is 40 m will be the hypotenuse and the angle between the sides AB and AC will be 90°.

By using Formula I, we can check that the area of the park is $\frac{1}{2} \times 32 \times 24$ m^2 = 384 m^2.

We find that the area we have got is the same as we found by using Heron's formula.

Now using Heron's formula, you verify this fact by finding the areas of other triangles discussed earlier *viz.*,

(i) equilateral triangle with side 10 cm.

(ii) isosceles triangle with unequal side as 8 cm and each equal side as 5 cm.

You will see that

For (i), we have $s = \frac{10 + 10 + 10}{2}$ cm = 15 cm.

Area of triangle $= \sqrt{15(15-10)(15-10)(15-10)}$ cm^2

$$= \sqrt{15 \times 5 \times 5 \times 5}\ \text{cm}^2 = 25\sqrt{3}\ \text{cm}^2$$

For (ii), we have $s = \frac{8 + 5 + 5}{2}$ cm = 9 cm

Area of triangle $= \sqrt{9(9-8)(9-5)(9-5)}$ cm^2 $= \sqrt{9 \times 1 \times 4 \times 4}$ cm^2 = 12 cm^2.

Let us now solve some more examples:

Example 1 : Find the area of a triangle, two sides of which are 8 cm and 11 cm and the perimeter is 32 cm (see Fig. 12.6).

Solution : Here we have perimeter of the triangle = 32 cm, a = 8 cm and b = 11 cm.

Third side c = 32 cm – (8 + 11) cm = 13 cm

So, $2s = 32$, i.e., $s = 16$ cm,

$s - a = (16 - 8)$ cm = 8 cm,

$s - b = (16 - 11)$ cm = 5 cm,

$s - c = (16 - 13)$ cm = 3 cm.

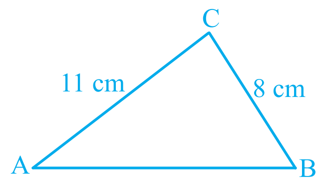

Fig. 12.6

Therefore, area of the triangle = $\sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{16 \times 8 \times 5 \times 3}\ \text{cm}^2 = 8\sqrt{30}\ \text{cm}^2$$

Example 2 : A triangular park ABC has sides 120m, 80m and 50m (see Fig. 12.7). A gardener *Dhania* has to put a fence all around it and also plant grass inside. How much area does she need to plant? Find the cost of fencing it with barbed wire at the rate of ₹20 per metre leaving a space 3m wide for a gate on one side.

Solution : For finding area of the park, we have

$2s$ = 50 m + 80 m + 120 m = 250 m.

i.e., s = 125 m

Now, $s - a = (125 - 120)$ m = 5 m,

$s - b = (125 - 80)$ m = 45 m,

$s - c = (125 - 50)$ m = 75 m.

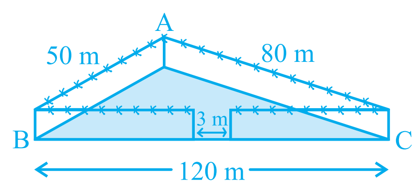

Fig. 12.7

Therefore, area of the park = $\sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{125 \times 5 \times 45 \times 75}\ \text{m}^2$$

$$= 375\sqrt{15}\ \text{m}^2$$

Also, perimeter of the park = AB + BC + CA = 250 m

Therefore, length of the wire needed for fencing = 250 m – 3 m (to be left for gate)

= 247 m

And so the cost of fencing = ₹20 × 247 = ₹4940

Example 3 : The sides of a triangular plot are in the ratio of 3 : 5 : 7 and its perimeter is 300 m. Find its area.

Solution : Suppose that the sides, in metres, are $3x$, $5x$ and $7x$ (see Fig. 12.8).

Then, we know that $3x + 5x + 7x = 300$ (perimeter of the triangle)

Therefore, $15x = 300$, which gives $x = 20$.

So the sides of the triangle are 3×20 m, 5×20 m and 7×20 m

i.e., 60 m, 100 m and 140 m.

Can you now find the area [Using Heron's formula]?

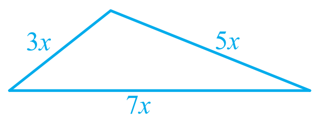

Fig. 12.8

We have $s = \dfrac{60 + 100 + 140}{2}$ m = 150 m,

and area will be $\sqrt{150(150-60)(150-100)(150-140)}$ m^2

$= \sqrt{150 \times 90 \times 50 \times 10}$ m^2

$= 1500\sqrt{3}$ m^2

EXERCISE 12.1

1. A traffic signal board, indicating 'SCHOOL AHEAD', is an equilateral triangle with side 'a'. Find the area of the signal board, using Heron's formula. If its perimeter is 180 cm, what will be the area of the signal board?
2. The triangular side walls of a flyover have been used for advertisements. The sides of the walls are 122 m, 22 m and 120 m (see Fig. 12.9). The advertisements yield an earning of ₹ 5000 per m^2 per year. A company hired one of its walls for 3 months. How much rent did it pay?

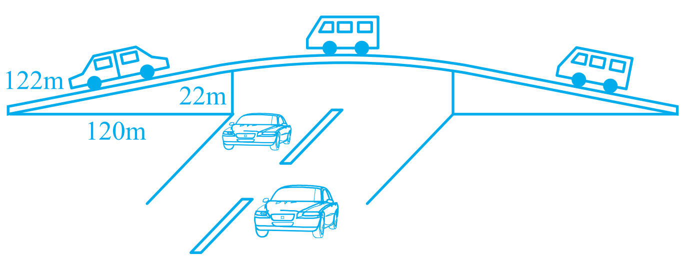

Fig. 12.9